

Final Examination, 10 May 1999
SM311O (Spring 1999) (Solutions)

The following formulas may be useful to you:

$$a) \oint_C \mathbf{v} \cdot d\mathbf{r} = \int \int_S \nabla \times \mathbf{v} \cdot d\mathbf{A},$$

$$b) \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \cdot \mathbf{v} \right) = -\nabla p + \mu \Delta \mathbf{v} + \rho \mathbf{F}, \quad \text{div } \mathbf{v} = 0.$$

$$c) \quad -fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} + A_v \frac{\partial^2 u}{\partial z^2}, \quad fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + A_v \frac{\partial^2 v}{\partial z^2}, \quad 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g.$$

1. (a) Let $\mathbf{v} = \langle xy, a \sin(zy), z\sqrt{x} \rangle$ where a is a constant. Determine a so that the divergence of \mathbf{v} at the point $P = (4, -1, \frac{\pi}{6})$ vanishes.

Solution: $\text{div}(\langle xy, a \sin(zy), z\sqrt{x} \rangle) = \frac{\partial(xy)}{\partial x} + \frac{\partial(a \sin(zy))}{\partial y} + \frac{\partial(z\sqrt{x})}{\partial z} = \sqrt{x} + y + a \cos(yz)$. Evaluating this expression at $P = (4, -1, \frac{\pi}{6})$ and solving the result for a leads to $a = \frac{-4\sqrt{3}}{\pi}$.

- (b) $\mathbf{v} = \langle y^2, -x^2, 0 \rangle$. Find the curl of \mathbf{v} . Is this flow irrotational anywhere?

Solution: $\nabla \times \langle y^2, -x^2, 0 \rangle = \langle 0, 0, -2x - 2y \rangle$. The flow is irrotational along the plane $x + y = 0$.

- (c) Prove the identity $\nabla \times \nabla \phi = \mathbf{0}$ if ϕ is an arbitrary function of x , y , and z .

Solution: $\nabla \times \nabla \phi = \nabla \times (\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \rangle)$. The first component of this vector is $\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}$ which is zero for smooth functions. The other two components are zero similarly.

2. Verify by direct differentiation if

- (a) $u(z) = e^{2z} \cos 2z$ is a solution of $u'''' + a^2 u = 0$ for any a .

Solution: $u''''(z) = -64e^{2z} \cos 2z$. So $u'''' + a^2 u = 0$ if $a^2 = 64$ or if $a = \pm 8$.

- (b) $u(x, y) = \sin x \cos 2y$ is an eigenfunction of the Laplace operator $-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. What is the eigenvalue?

Solution: Let $u(x, y) = \sin x \cos 2y$. Then

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 5 \sin x \cos 2y.$$

So $\lambda = 5$.

3. (a) Give a parametrization for the plane that passes through the points $(-1, 1, 0)$, $(0, 2, 2)$, and $(1, 0, 1)$.

Solution: The equation of the plane is of the form $Ax + By + Cz = D$. Substituting the three points $(-1, 1, 0)$, $(0, 2, 2)$ and $(1, 0, 1)$ in the equation we get the following three equations in terms of A , B , C , and D :

$$(*) \quad -A + B = D, \quad 2B + 2C = D, \quad A + C = D.$$

Adding the first and the third equation leads to $B + C = 2D$ which, when compared with $2B + 2C = D$ (see $(*)$), results in $D = 0$. The relevant equations in $(*)$ now become $A = B$ and $C = -B$. So $x + y - z = 0$ is the equation of the plane. (Equivalently, we can find the equation of the plane by constructing two vectors parallel to the plane from the three given points. In this case, the two vectors are $\langle 1, 1, 2 \rangle$ and $\langle 1, -2, -1 \rangle$. Then the cross product of these vectors, $\langle 3, 3, -3 \rangle$, is normal to the plane.)

Finally, using the normal and the point $(1, 0, 1)$ we get $x + y - z = 0$ to be the equation of the plane, or equivalently, $z = x + y$. So

$$\mathbf{r}(u, v) = \langle u, v, u + v \rangle$$

is the parametrization of the plane.

- (b) Find a unit normal vector to the surface of the upper hemisphere of the Earth at the point whose longitude and latitude are 60 and 30 degrees, respectively.

Solution: $\mathbf{r}(u, v) = \langle a \cos u \cos v, a \sin u \cos v, a \sin v \rangle$, $u \in (0, 2\pi)$, $v \in (-\frac{\pi}{2}, \frac{\pi}{2})$, is the parametrization of the upper hemisphere of a sphere of radius a . A normal to this surface at any point parametrized by u and v is given by $\mathbf{r}_u \times \mathbf{r}_v$ which equals

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \langle -a \sin u \cos v, a \cos u \cos v, 0 \rangle \times \langle -a \cos u \sin v, -a \sin u \sin v, a \cos v \rangle =$$

$$a^2 \cos v \langle \cos u \sin v, \sin u \sin v, \cos v \rangle.$$

A unit vector in the direction of \mathbf{N} is

$$\langle \cos u \cos v, \sin u \cos v, \sin v \rangle,$$

which, when evaluated at $u = \frac{\pi}{3}$ and $v = \frac{\pi}{6}$, yields

$$\mathbf{n} = \left\langle \frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{1}{2} \right\rangle.$$

4. (a) The function $\phi(x, y) = ax^2y^2 - by^2 + x$ is the potential for a velocity vector field \mathbf{v} . Determine the values of a and b so that the velocity of the particle located at $(2, -1, 1)$ is zero.

Solution: Because $\mathbf{v} = \nabla\phi$ we have

$$\mathbf{v} = \langle 2axy^2 + 1, 2ax^2y - 2by \rangle$$

Substitute $(2, -1)$ into \mathbf{v} :

$$\mathbf{v}|_{(2,-1)} = \langle 1 + 4a, -8a + 2b \rangle$$

which is $\mathbf{0}$ if $a = -\frac{1}{4}$ and $b = -1$.

- (b) The function $\psi(x, y) = ax^2 + xy + by^2$ is the stream function of a velocity field \mathbf{v} . Find a and b so that the velocity of the particle located at $(1, 1)$ has magnitude $\frac{1}{2}$.

Solution: Since $\psi = ax^2 + xy + by^2$, then

$$\mathbf{v} = \langle x + 2by, -2ax - y \rangle.$$

Now, the velocity at $(1, 1)$ is $\mathbf{v}|_{(1,1)} = \langle 1 + 2b, -2a - 1 \rangle$. After setting the magnitude of this vector equal to $\frac{1}{2}$ we get that $(1 + 2b)^2 + (1 + 2a)^2 = \frac{1}{4}$.

5. (a) Consider the velocity field $\mathbf{v} = x^2z\mathbf{k}$. Determine the flux of this fluid through the following two surfaces:

- i. a disk of radius 1 in the xy -plane and centered at the origin.

Solution: The flux is zero since the integrand is zero on the xy -plane.

- ii. a disk of radius 1 in the plane $z = 1$ and centered at the origin.

Solution: The parametrization of the disk is

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 \rangle.$$

So $\mathbf{r}_u \times \mathbf{r}_v = u\mathbf{k}$. Hence

$$\text{Flux} = \int_0^1 \int_0^{2\pi} u^3 \cos^2 v \, dv du = \frac{\pi}{4}.$$

- (b) Compute the flux of vorticity of $\mathbf{v} = y^2\mathbf{i}$ through the surface of the upper hemisphere of a sphere of radius 2 centered at the origin (Hint: Use the Stokes Theorem).

Solution: By the Stokes theorem, the flux in question is equal to $\int_C \mathbf{v} \cdot d\mathbf{r}$ where C is a circle of radius 2 in the xy -plane. Parametrizing this circle as $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 0 \rangle$, we see that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = - \int_0^{2\pi} 8 \sin^3 t \, dt = 0.$$

6. Consider the following heat conduction problem:

$$u_t = 7u_{xx}, \quad u(0, t) = u(3, t) = 0, \quad u(x, 0) = x(3 - x).$$

- (a) Use separation of variables and find the solution to this problem. Clearly indicate the process of separation of variables and the Fourier Series method used in obtaining this solution.

Solution: After using separation of variables, whose detail I will not elaborate on here, we get

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\frac{7n^2\pi^2 t}{9}} \sin \frac{n\pi x}{3}.$$

The coefficients A_n are determined from

$$A_n = \frac{(u(x, 0), \sin \frac{n\pi x}{3})}{(\sin \frac{n\pi x}{3}, \sin \frac{n\pi x}{3})}.$$

Using a calculator, we find that

$$A_1 = 2.32211, \quad A_2 = 0, \quad A_3 = 0.0860041.$$

- (b) Use the first nonzero term of the above solution and estimate how long it takes for the temperature at $x = 1.5$ to reach 50 per cent of its original value.

Solution: Using only the first term of the Fourier Series solution, we have

$$u(x, t) = 2.32211 e^{-\frac{7\pi^2 t}{9}} \sin \frac{\pi x}{3}.$$

Now, $u(\frac{3}{2}, 0) = \frac{9}{4}$. So our problem reduces to determining t so that

$$2.32211 e^{-\frac{7\pi^2 t}{9}} = \frac{9}{8}.$$

It turns out that $t = 0.0944059$.

7. Let $\mathbf{v} = \langle x^2 + y^2, 2xy \rangle$ be the velocity field of a fluid. Compute the acceleration \mathbf{a} of this flow. Does \mathbf{a} have a potential p ? If yes, find it.

Solution: $\mathbf{a} = \mathbf{v} \cdot \nabla \mathbf{v} = \langle 2(x^3 + 3xy^2), 2(3x^2y + y^3) \rangle$. The curl of this vector is $\mathbf{0}$ so the acceleration vector has a potential ϕ . Since

$$(**) \quad \frac{\partial \phi}{\partial x} = 2x^3 + 6xy^2, \quad \frac{\partial \phi}{\partial y} = 6x^2y + 2y^3,$$

we start by integrating the first equation with respect to x to get

$$\phi = \frac{1}{2}x^4 + 3x^2y^2 + f(y).$$

Next, we differentiate the second equation with respect to y and compare it with the second equation in $(**)$ to get that $f'(y) = 2y^3$ or $f(y) = \frac{1}{2}y^4$. So $\phi = \frac{1}{2}x^4 + 3x^2y^2 + \frac{1}{2}y^4$.

8. Let $\mathbf{\Omega}$ stand for the angular velocity of our planet.

- (a) Noting that our planet rotates once every 24 hours, compute Ω where $\mathbf{\Omega} = \langle 0, 0, \Omega \rangle$. What are the units of Ω ?

Solution: $\Omega = \frac{2\pi}{24 \times 3600} = 0.0000727221 \text{ rad/sec.}$

- (b) Use this value of $\mathbf{\Omega}$ and estimate the values in the centripetal acceleration $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ where \mathbf{r} is the position vector to a typical point on the surface of the Earth. Assume that the radius of the Earth is 6000 kilometers.

Solution: Let $\mathbf{r} = \langle x, y, z \rangle$. Then

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\langle \Omega^2 x, \Omega^2 y, 0 \rangle$$

the terms $\Omega^2 x$ and $\Omega^2 y$ take on the largest values when x or y is 6000 kilometers. With the value of Ω found previously, $\Omega^2 x \leq 0.031731$ meters per second per second, much smaller than 9.8 meters per second per second, the acceleration of gravity.

9. Consider an incompressible fluid occupying the basin

$$D = \{(x, y, z) | 0 \leq z \leq H\}.$$

Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be the velocity field of a motion generated in D . Suppose that we have been able to determine that

$$v_1(x, y, z) = x^2 y^2, \quad v_2(x, y, z) = -3x^2 z,$$

but have only succeeded in measuring v_3 at the bottom of the basin and that this value is

$$v_3(x, y, 0) = x + y.$$

Determine v_3 everywhere in D . (Hint: What does incompressibility mean **mathematically**?)

Solution: From the equation of incompressibility we have

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0$$

or

$$\frac{\partial v_3}{\partial z} = -\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y}.$$

Substituting the values of v_1 and v_2 in the above relation yields

$$\frac{\partial v_3}{\partial z} = -2xy^2.$$

Integrating this result with respect to z from 0 to z and using the value of v_3 at $z = 0$ yields

$$v_3 = -2xy^2 z + x + y.$$

10. A flow is called geostrophic if the velocity $\mathbf{v} = \langle u(x, y), v(x, y) \rangle$ and the pressure gradient ∇p are related by

$$(\ast \ast \ast) \quad -fv = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad fu = -\frac{1}{\rho} \frac{\partial p}{\partial y},$$

where ρ , a constant, is the density of the fluid, and f is the coriolis parameter.

- (a) Assuming that f is constant, prove that the divergence of \mathbf{v} must vanish.

Solution: From the equations of motion we have

$$u = -\frac{1}{\rho f} \frac{\partial p}{\partial y}, \quad v = \frac{1}{\rho f} \frac{\partial p}{\partial x}.$$

Now $\text{div } \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$ which is equal to

$$-\frac{1}{\rho f} \frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho f} \frac{\partial^2 p}{\partial y \partial x} = 0$$

when p is a smooth function.

- (b) Prove that the particle paths of a geostrophic flow and its isobars coincide.

Solution: Note that $\mathbf{v} \cdot \nabla p = -\frac{1}{\rho f} \frac{\partial p}{\partial y} \frac{\partial p}{\partial x} + \frac{1}{\rho f} \frac{\partial p}{\partial x} \frac{\partial p}{\partial y} = 0$. So \mathbf{v} and ∇p are orthogonal. Since ∇p is orthogonal to isobars, and since \mathbf{v} is tangential to particle paths, particle paths and isobars coincide.

- (c) Consider a high pressure field in a geostrophic flow in the northern hemisphere (where $f > 0$). By appealing to the equations in $(\ast \ast \ast)$ explain whether this high pressure field results in a clockwise or a counterclockwise motion.

Solution: Without loss of generality, assume that the high pressure occurs at the origin of the coordinate system. Let P be a point in the first quadrant. Then ∇p at P points toward the origin because 0 is a maximum of p . Then $\frac{\partial p}{\partial x} \leq 0$ and $\frac{\partial p}{\partial y} \leq 0$ at P (draw a picture to convince yourself of this). Going back to the geostrophic equations, $u \geq 0$ and $v \leq 0$ at P which indicates that the motion is clockwise.